1(a) Let $\alpha > -1$ and f a monotone function on $(0, 1]$. If the improper integral $\int_0^1 x^\alpha f(x)\,dx$ exists, then $\lim_{x\to 0} x^{\alpha+1} f(x) = 0$

Pf: If $\lim_{x\to 0} f(x) \neq \pm\infty$, then $\lim_{x\to 0} x^{\alpha+1} f(x) = 0$ is trivial

W.L.O.G., suppose $\lim_{x\to 0} f(x) = \infty$. Then f is non-increasing and $\exists\ \delta' > 0$ s.t. $f(x) > 0$, $\forall\ 0 < x < \delta'$.

Fix $\varepsilon > 0$. Since $\int_0^1 x^\alpha f(x)\,dx$ exists, by Cauchy's Criterion, there exists $\delta \in (0, \delta')$ s.t. for any $0 < x_1 < x_2 < \delta$, $\int_{x_1}^{x_2} x^\alpha f(x)\,dx < \frac{\varepsilon}{\alpha+1}\left(1 - \frac{1}{2^{\alpha+1}}\right)$

For any $y \in (0, \delta)$, let $x_1 = \frac{y}{2}$ and $x_2 = y$.

Then
$$\begin{aligned}
\frac{\varepsilon}{\alpha+1}\left(1-\frac{1}{2^{\alpha+1}}\right) &> \int_{\frac{y}{2}}^{y} x^\alpha f(x)\,dx \\
&> \int_{\frac{y}{2}}^{y} x^\alpha f(y)\,dx \\
&= f(y) \int_{\frac{y}{2}}^{y} x^\alpha\,dx \\
&= \frac{f(y)}{\alpha+1} x^{\alpha+1} \Big|_{\frac{y}{2}}^{y} \\
&= \frac{1}{\alpha+1}\left(1-\frac{1}{2^{\alpha+1}}\right) y^{\alpha+1} f(y)
\end{aligned}$$

Therefore, for any $y \in (0, \delta)$, $y^{\alpha+1} f(y) < \varepsilon$.

Hence, $\lim_{x\to 0} x^{\alpha+1} f(x) = 0$ $\square$

(b) If $f \in R[0,1]$, then the set of all continuous points is dense on $[0,1]$.

If f is a function on $\mathbb{R}$ with $\int_{-\infty}^{\infty} f$ exists, the statement still holds.

Pf: • Suppose not.

Then there exists an open interval I such that f is discontinuous at any $x \in I$.

Since $f \in R[0,1]$, by Lebesgue Criterion, the set D of all discontinuous points is a null set.

Pick $\varepsilon = |I|$.

By def, there exists a countable collection $\{J_n\}_{n=1}^{\infty}$ of open intervals such that $\bigcup_{n=1}^{\infty} J_n \supset D \supset I$ and

$$\sum_{n=1}^{\infty} |J_n| < \varepsilon = |I|.$$

Contradiction!

Suppose f is on $\mathbb{R}$ and $\int_{-\infty}^{\infty} f$ exists.

Let $E = \{x \in \mathbb{R} : f \text{ continuous at } x\}$.

Pick any nonempty open $U \subset \mathbb{R}$, we wish to show $U \cap E \neq \phi$.

Pick M large such that $U \cap (-M, M) \neq \phi$.

Let $V = U \cap (-M, M)$.

Since $\int_{-\infty}^{\infty} f$ exists, $f|_{[-M,M]} \in R[-M, M]$.

Then $E \cap [-M, M]$ is dense on $[-M, M]$.

Thus $E \cap [-M, M] \cap V \neq \phi$.

Hence $U \cap E \neq \phi$.

$\square$

(c) Let $f \in R[0,1]$. Then $\int_0^1 f^2 = 0$ if and only if $f(c) = 0$ for all continuous point c of f.

Pf: ($\Rightarrow$) Suppose there exists $c \in [0,1]$ s.t. f is continuous at c and $f(c) \neq 0$.

Then f^2 is continuous at c and $f^2(c) > 0$.

By continuity, there exists an interval $[a,b] \subset [0,1]$ such that $c \in [a,b]$ and $f^2(x) \geq \frac{f^2(c)}{2}$ for any $x \in [a,b]$

Thus $\int_0^1 f^2(x)dx \geq \int_a^b f^2(x)\,dx$

$$\geq (b-a)\frac{f^2(c)}{2} > 0.$$

Contradiction!

($\Leftarrow$) Suppose $f(c) = 0$ for all continuous points c of f.

By (b), f is continuous on a dense set of $[0,1]$.

Then $f=0$ on a dense set of $[0,1]$.

Pick any partition $P=(x_0, x_1, \dots, x_n)$ of $[0,1]$.

$$L(f;P) = \sum_{n=1}^{\infty} \inf\{f(x) : x\in[x_{n-1}, x_n]\}(x_n - x_{n-1})$$

$$= \sum_{n=1}^{\infty} 0\cdot(x_n - x_{n-1})$$

$$= 0.$$

Since P is arbitrary, $\underline{\int_0^1} f = L(f)$

$$= \sup_P L(f;P)$$

$$= 0.$$

Since $f\in R[0,1]$, $\int_0^1 f = \underline{\int_0^1} f = 0$.

$\square$

2. $u: \mathbb{R}^n \to \mathbb{R}$ satisfies for any $\vec{x}, \vec{y} \in \mathbb{R}^n$, $t > 0$

- $u(\vec{x}+\vec{y}) \leq u(\vec{x}) + u(\vec{y})$
- $u(t\vec{x}) = t\,u(\vec{x})$

(a) Fix $\vec{x}, \vec{h} \in \mathbb{R}$. Show that

(i) $f: \mathbb{R} \to \mathbb{R}$ by $f(t) = u(\vec{x} + t\vec{h})$ is convex.

(ii) $g: \mathbb{R}\setminus\{0\} \to \mathbb{R}$ by $g(t) = \dfrac{u(\vec{x}+t\vec{h}) - u(\vec{x})}{t}$ is monotone.

Pf: (i) For any $t, s \in \mathbb{R}$, for any $\alpha \in (0,1)$,

$$\begin{aligned}
f(\alpha t + (1-\alpha)s) &= u(\vec{x} + [\alpha t + (1-\alpha)s]\vec{h}) \\
&= u(\alpha\vec{x} + (1-\alpha)\vec{x} + \alpha t\vec{h} + (1-\alpha)s\vec{h}) \\
&= u\big(\alpha(\vec{x}+t\vec{h}) + (1-\alpha)(\vec{x}+s\vec{h})\big) \\
&\leq u(\alpha(\vec{x}+t\vec{h})) + u((1-\alpha)(\vec{x}+s\vec{h})) \\
&= \alpha\, u(\vec{x}+t\vec{h}) + (1-\alpha)\, u(\vec{x}+s\vec{h}) \\
&= \alpha f(t) + (1-\alpha) f(s).
\end{aligned}$$

(ii) Here, we assume $0 < t < s$

The other cases ($t < s < 0$ / $t < 0 < s$) are left as an exercise.

$$\begin{aligned}
g(s) - g(t) &= \frac{u(\vec{x}+s\vec{h}) - u(\vec{x})}{s} - \frac{u(\vec{x}+t\vec{h}) - u(\vec{x})}{t} \\
&= \frac{t\,u(\vec{x}+s\vec{h}) - t\,u(\vec{x}) - s\boxed{u(\vec{x}+t\vec{h})} + s\,u(\vec{x})}{st}
\end{aligned}$$

$$\left(t = \frac{t}{s}\cdot s + \left(1-\frac{t}{s}\right)\cdot 0 \right) \qquad \geq \frac{tu(\vec{x}+s\vec{h}) - tu(\vec{x}) - s\left(\boxed{\frac{t}{s}u(\vec{x}+s\vec{h}) + \left(1-\frac{t}{s}\right)u(\vec{x})}\right) + su(\vec{x})}{st}$$

$$= 0$$

$\square$

(b) Fix $\vec{x}, \vec{h} \in \mathbb{R}^n$. Show that

$$\lim_{t\to 0} \frac{u(\vec{x}+t\vec{h}) - u(\vec{x})}{t} \text{ exists iff } \lim_{t\to 0} \frac{u(\vec{x}+t\vec{h}) + u(\vec{x}-t\vec{h}) - 2u(\vec{x})}{t} = 0$$

Pf: ($\Rightarrow$) Suppose $c := \lim_{t\to 0} \dfrac{u(\vec{x}+t\vec{h}) - u(\vec{x})}{t}$ exists.

Then
$$\lim_{t\to 0} \frac{u(\vec{x}+t\vec{h}) + u(\vec{x}-t\vec{h}) - 2u(\vec{x})}{t}$$
$$= \lim_{t\to 0} \frac{u(\vec{x}+t\vec{h}) - u(\vec{x})}{t} - \lim_{t\to 0} \frac{u(\vec{x}-t\vec{h}) - u(\vec{x})}{-t}$$
$$= c - c = 0$$

($\Leftarrow$) Suppose
$$a := \liminf_{t\to 0} \frac{u(\vec{x}+t\vec{h}) - u(\vec{x})}{t} < \limsup_{t\to 0} \frac{u(\vec{x}+t\vec{h}) - u(\vec{x})}{t} =: b.$$

Then
$$\limsup_{t\to 0} \frac{u(\vec{x}+t\vec{h}) + u(\vec{x}-t\vec{h}) - 2u(\vec{x})}{t}$$
$$= \limsup_{t\to 0} \frac{u(\vec{x}+t\vec{h}) - u(\vec{x})}{t} - \liminf_{t\to 0} \frac{u(\vec{x}-t\vec{h}) - u(\vec{x})}{-t}$$

$= b - a > 0$.

Contradiction!

$\underline{17}$

(c) Fix $\vec{x} \in \mathbb{R}^n$. Suppose $u'_{\vec{x}}(\vec{h}) = \lim_{t \to 0} \frac{u(\vec{x}+t\vec{h}) - u(\vec{x})}{t}$ exists for all $\vec{h} \in \mathbb{R}^n$.

Show that $u'_{\vec{x}} : \mathbb{R}^n \to \mathbb{R}$ is linear.

Pf: • Pick any $\alpha \in \mathbb{R}$.

If $\alpha = 0$, $u'_{\vec{x}}(\alpha\vec{h}) = 0 = \alpha u'_{\vec{x}}(\vec{h})$

If $\alpha \neq 0$, $u'_{\vec{x}}(\alpha\vec{h}) = \lim_{t \to 0} \frac{u(\vec{x}+t\alpha\vec{h}) - u(\vec{x})}{t}$

$$= \alpha \lim_{t \to 0} \frac{u(\vec{x}+t\alpha\vec{h}) - u(\vec{x})}{t\alpha}$$

$$= \alpha \lim_{s \to 0} \frac{u(\vec{x}+s\vec{h}) - u(\vec{x})}{s} \qquad (s = t\alpha)$$

$$= \alpha u'_{\vec{x}}(\vec{h}).$$

• Pick any $\vec{h}, \vec{h}' \in \mathbb{R}^n$.

Then $u\left(\vec{x} + \frac{t}{2}(\vec{h}+\vec{h}')\right) = \frac{1}{2} u\left(2\vec{x} + t(\vec{h}+\vec{h}')\right)$

$$\leq \frac{1}{2}\left(u(\vec{x}+t\vec{h}) + u(\vec{x}+t\vec{h}')\right)$$

For $t > 0$, $\frac{u\left(\vec{x}+\frac{t}{2}(\vec{h}+\vec{h}')\right)}{\frac{t}{2}} \leq \frac{u(\vec{x}+t\vec{h})}{t} + \frac{u(\vec{x}+t\vec{h}')}{t}$.

Since $u_{\vec{x}}(\vec{h}) := \lim_{s \to 0} \frac{u(\vec{x}+s\vec{h})}{s}$ exists for any $\vec{h} \in \mathbb{R}^n$,

$$u_{\vec{x}}(\vec{h}+\vec{h}') = \lim_{t\to 0^+} \frac{u(\vec{x}+t(\vec{h}+\vec{h}')) - u(\vec{x})}{t} \le \lim_{t\to 0^+} \frac{u(\vec{x}+t\vec{h})}{t} + \lim_{t\to 0^+} \frac{u(\vec{x}+t\vec{h}')}{t}$$

$$= u_{\vec{x}}(\vec{h}) + u_{\vec{x}}(\vec{h}').$$

For $t<0$, $\dfrac{u\left(\vec{x}+\frac{t}{2}(\vec{h}+\vec{h}')\right)}{\frac{t}{2}} \ge \dfrac{u(\vec{x}+t\vec{h})}{t} + \dfrac{u(\vec{x}+t\vec{h}')}{t}$

Thus $u_{\vec{x}}(\vec{h}+\vec{h}') = \lim_{t\to 0^-} \dfrac{u(\vec{x}+t(\vec{h}+\vec{h}')) - u(\vec{x})}{t} \ge \lim_{t\to 0^-} \dfrac{u(\vec{x}+t\vec{h})}{t} + \lim_{t\to 0^-} \dfrac{u(\vec{x}+t\vec{h}')}{t}$

$$= u_{\vec{x}}(\vec{h}) + u_{\vec{x}}(\vec{h}').$$

Hence, $u_{\vec{x}}(\vec{h}+\vec{h}') = u_{\vec{x}}(\vec{h}) + u_{\vec{x}}(\vec{h}')$.

$\square$